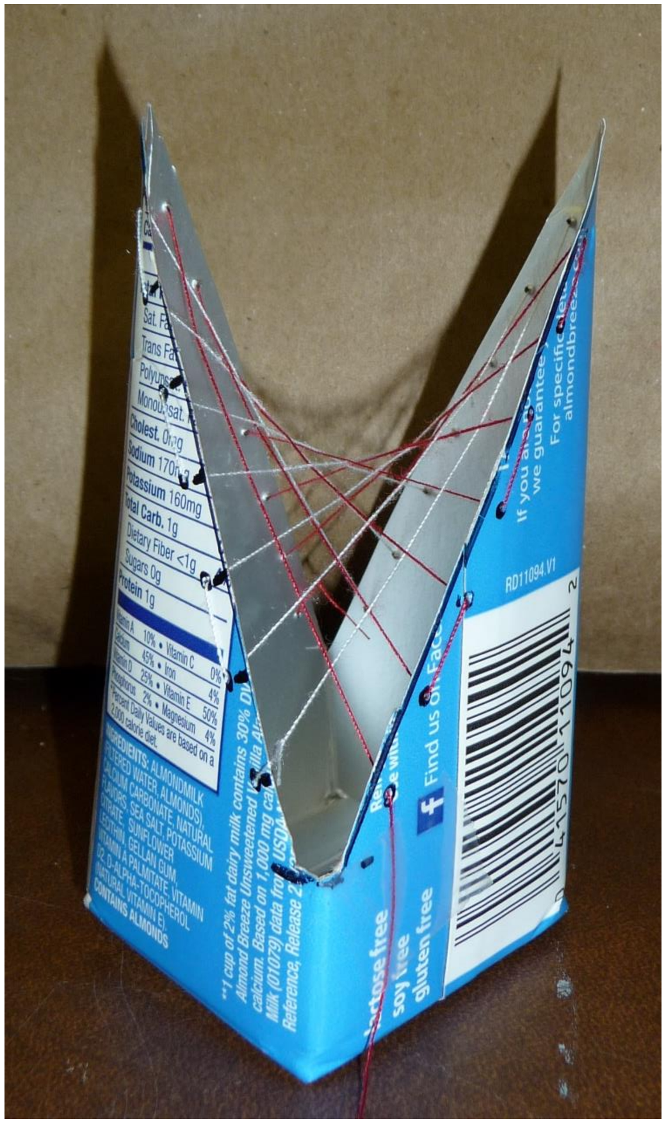
Saddle Geometry



By Joel Cryer

Topics: mathematics, planes, weaving, construction, architecture, paraboloid

Related Disciplines: mathematics, architecture

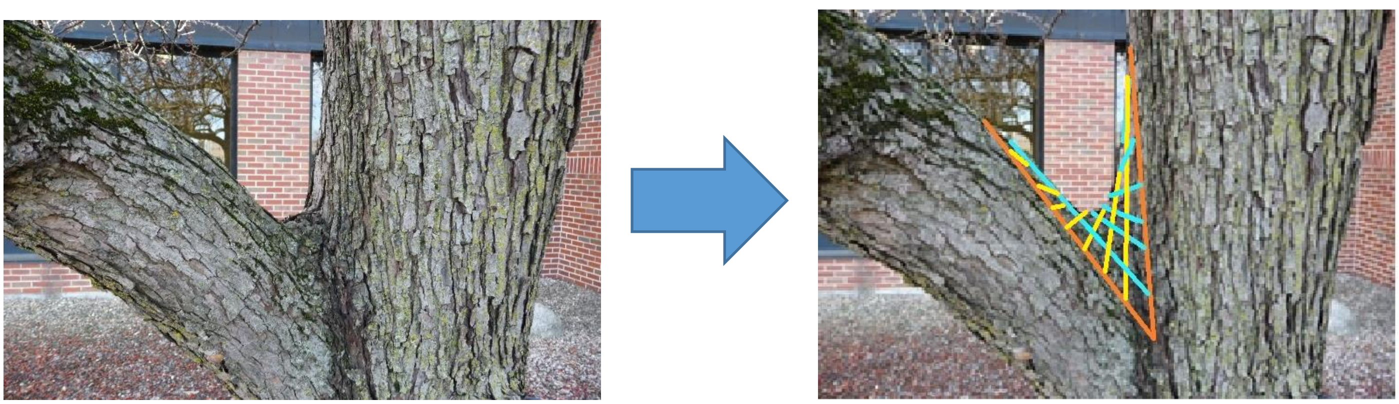
Objective: Construct a mathematically and aesthetically unique shape that is visible throughout nature.

Lesson: Part One: Introduction (20 minutes)

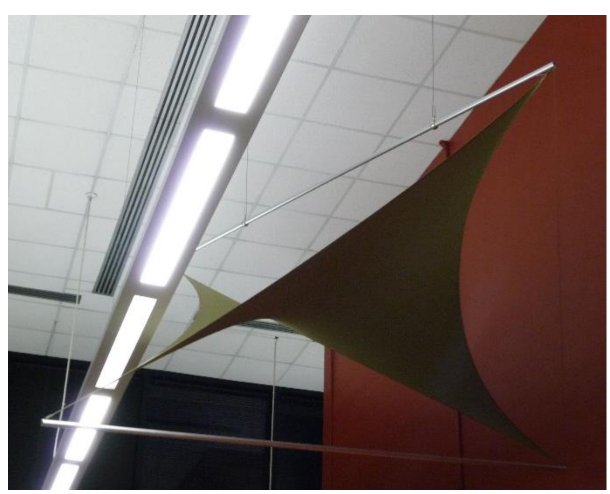
Saddle points are a prime example of art in mathematics. Their sweeping curves evoke a bird in flight, an ocean-dwelling animal, or some sort of alien spacecraft. One particular form of saddle point, the hyperbolic paraboloid, is the focus of this lesson.

Building your own paraboloid is easy, and its structure is intuitively understood as soon as you've built one, but its beauty and symmetry belie that simplicity.

Hyperbolic paraboloids and similar plane shapes are visible in nature, for example in the structures of certain flowers or in the crooks of tree branches:



The hyperbolic paraboloid has also been the inspiration for many forms of art, functional and otherwise. The famous Waikikian Hotel in Hawaii has a dramatically curved hyperbolic paraboloid roof. If you are in Madison, Wisconsin as we are, Van Hise Hall on the UW campus has some paraboloid artworks in the high-ceiling study room, room 455:



Lesson: Part Two: Class Project (60 minutes)

In this activity, students will build their own hyperbolic paraboloids out of cardboard and string. As written, this activity requires a needle and thread. If the needles prove hazardous, they could hypothetically be replaced with a hole punch, but be sure that the cartons available are large enough to allow this change. There needs to be enough space on the box surface that the holes will not interfere with each other.

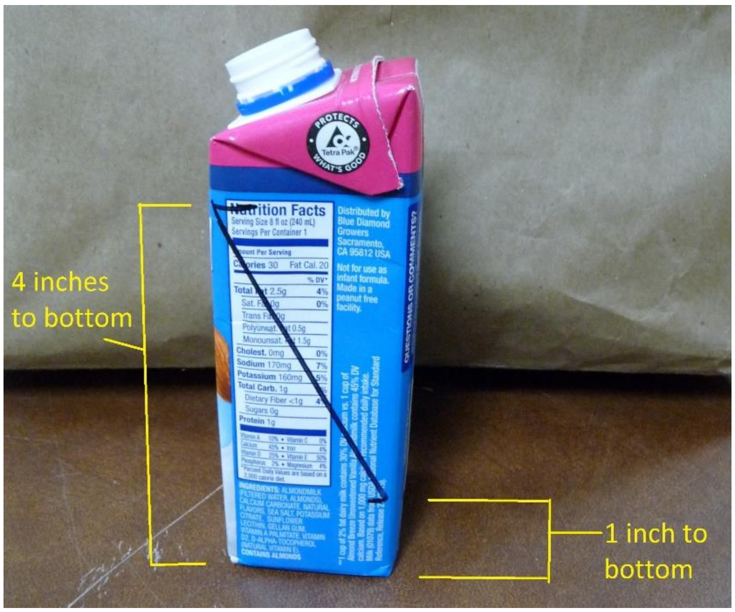
Materials:

- Square-base cartons, such as creamer or milk. Cartons are handy in that they have a premade square cross-section which makes for nice precise paraboloids.

The activity is adaptable to use sheet cardboard or non-square boxes if cartons are not available. However, cartons are recommended as sheet cardboard requires a lot of tangentially related prep work to achieve the same result as a stock carton.

- Hand sewing needles (can be large, this is not highly precise)
- Thread, preferably of at least two colors to denote the skewing pattern
- Ruler
- Permanent marker
- Scissors

1. Using the intact carton, set up diagonal lines on each face. For structural integrity the diagonals should not go all the way to the bottom of the carton, but rather to a predetermined distance from the bottom. On the carton illustrated, the lowest end of the diagonal is an inch from the bottom and the high end is 4 inches up.



The measurements are arbitrary round numbers, but a greater vertical difference makes for a nicer paraboloid.

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2. Repeat the marking pattern working around the carton. The resulting marks should leave two “valleys” and two “peaks” with the peaks being 180 degrees opposite of each other.

Do not cut the carton yet as keeping it intact will help with structural integrity for later steps.

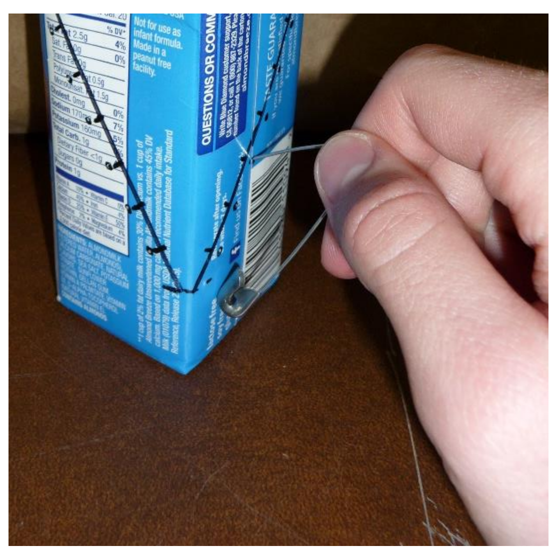
3. Measure how long the diagonal is. Then divide it evenly with a number of marks. I recommend at least 7 marks so that the paraboloid will be smoother, but the model should not be cluttered to the point of being hard to sew later.

Be sure students allow spaces at the extreme ends of the diagonal, they should not plan on n marks with the n^{th} mark falling at the very end. This does not need to be exact; it is okay if the diagonal did not come to a round length.

The diagonal resulting from our prior marking of the carton is about 3.5 inches long, and here has been divided so as to allow for 7 holes. There need to be 8 gaps for the 7 holes: 6 gaps between and a gap on each end of the diagonal. Therefore the marks are 3.5/8 = 7/16" apart.

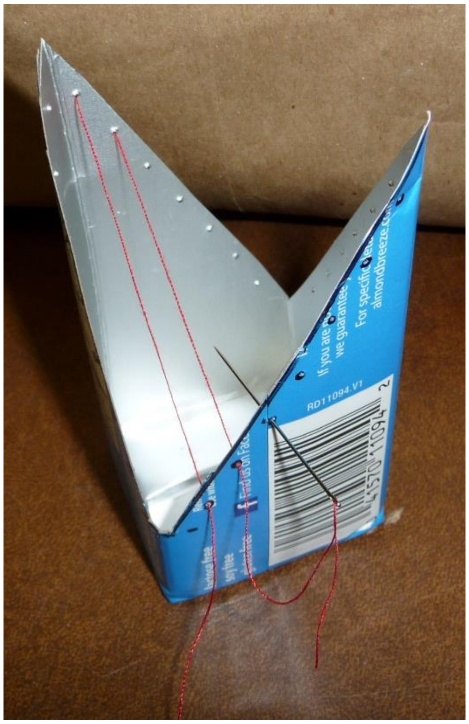
Note also that a second parallel line of round marks has been placed well off the diagonal, so that the diagonal can be cut without interfering with the marks.

4. Using a needle or safety pin, poke holes at each of the marks *not* centered on the diagonal.



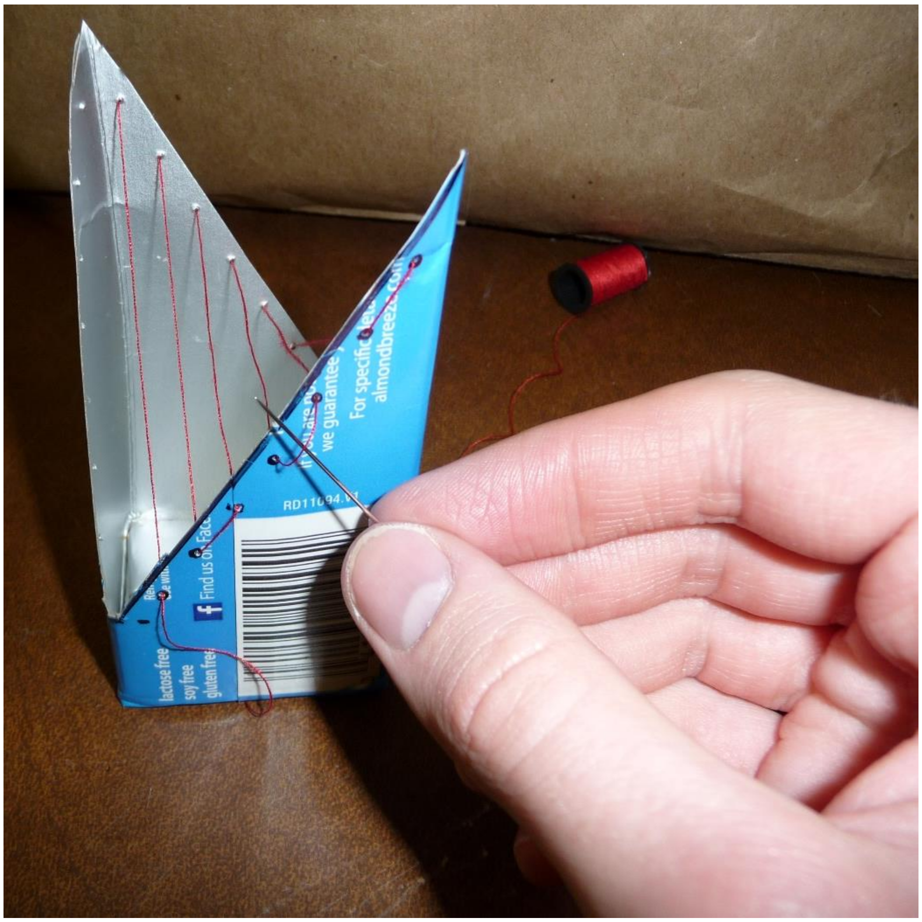
5. Cut along the diagonals drawn earlier to produce the final framework.

6. Using a needle and thread, connect opposite holes on two sides of the frame.



Upon reaching the final hole, tie off the thread onto the carton frame.

Important: After tying off the thread, work back through the carton, tensioning the thread at each loop outside the carton.

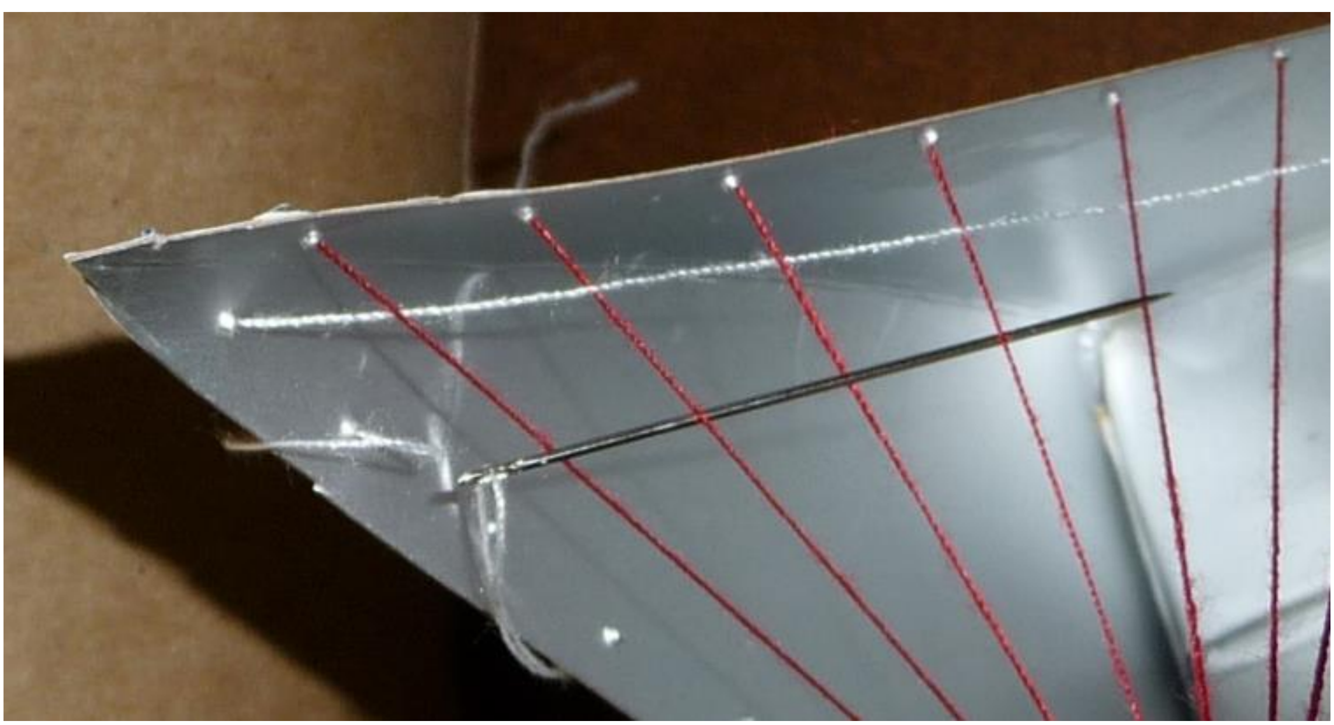


I recommend using the needle as shown to pull each loop until the two strings are taut inside the box. Then move on to the next loop and repeat down the line until you reach the first hole you threaded.

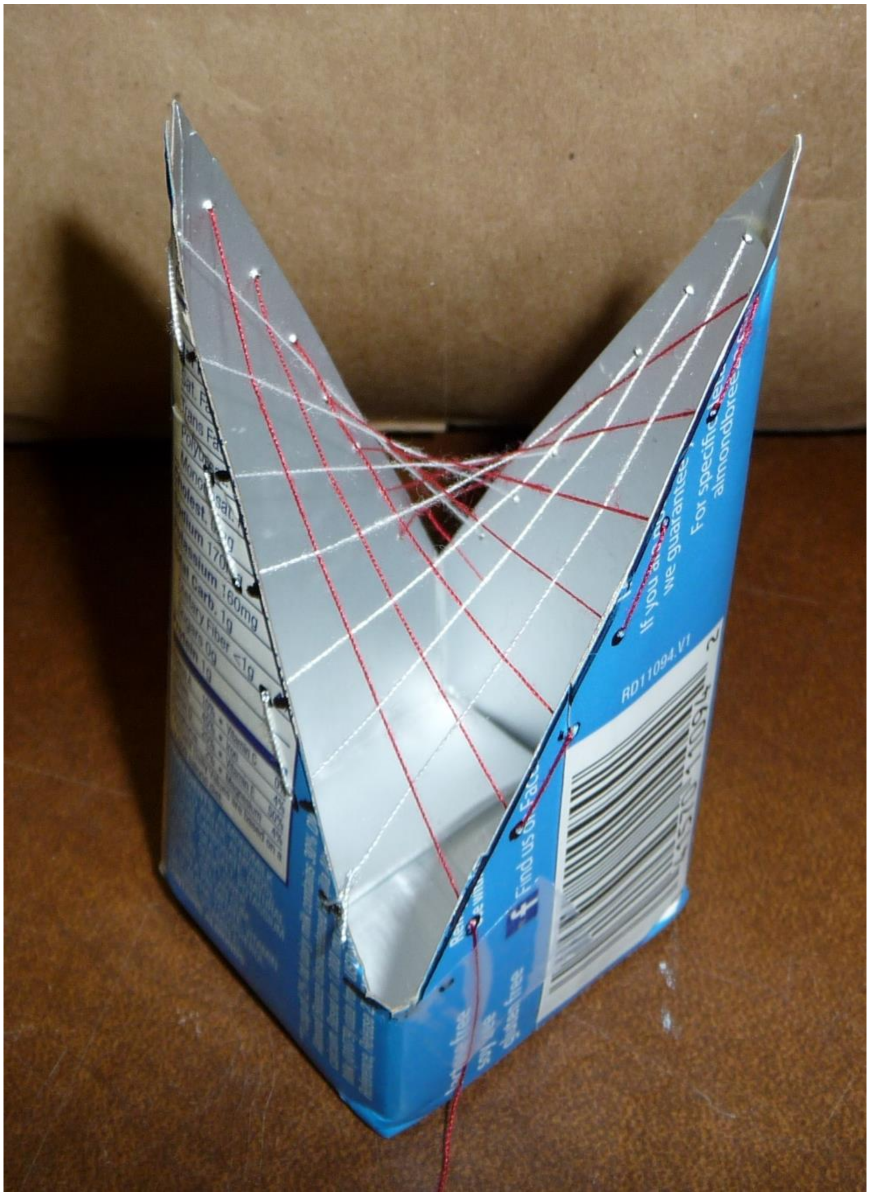
At the other end of the thread, tape the loose end to a side of the carton.

Note that this has already produced a warped plane; the paraboloid results from two such planes which interweave to become a single plane with a different curvature.

7. Using a different color of thread, repeat Step 6 for the other two sides. With these two sides, the string needs to be woven through the prior string:



You're done! Observe the woven shape that has resulted: a hyperbolic paraboloid.



Lesson: Part Three: Discussion

1. Have students name places in nature that they have observed this shape, aside from flowers and tree branches as previously mentioned.

2. What would happen if you flex the plane? What would the resulting motion look like? (There is a video cited in the references that answers this question, once it has been visualized and discussed.)

3. As noted in the introduction, hyperbolic paraboloids have occasionally been used in architecture. However, clearly the majority of buildings do not do this, and even those with unusual, curving roofs tend to curve in other ways (for instance, domes). Name a few structural reasons this might be so.

4. Once students have seen the shape in three dimensions, have them draw it in 2D by whatever method they like (repeated cross-sections, a single perspective view, etc.). Have them explain briefly why they visualized it that way – was it the most natural interpretation for them? The easiest to draw? Another reason?

Homework:

Find a non-square container and string its opposite sides (or approximate opposite sides) together, as you did the square top carton. Observe the shape that results in the string – how does it resemble or differ from the paraboloid?

This does not need to be mathematically precise by any means, just as close as you can be. Suggested materials include the cardboard cans that certain snacks or frozen juice come in, or a planar piece of thin cardboard (from a cereal box, etc.) that has been bent into a new shape.

Further Reading:

All pictures used in text are my own so the text has no citations per se. However, it is worth referencing several other sources as further reading:

http://www.thingiverse.com/thing:183812

This laser-cut piece was the original inspiration for these instructions. Many others from amateurs to professional artists have used similar methods; it is worth a Google Image search of "hyperbolic paraboloid string" to see the range of works.

http://www.math.tamu.edu/~frank.sottile/research/stories/hyperbolic_football/index.html

This site has patterns and explanations for the "hyperbolic football." The hyperbolic football resembles the net for a soccer ball but the pentagons are replaced by heptagons. That seemingly small change converts the net from folding nicely into a ball to folding erratically into a plane with many saddle points.

Having attempted to make the hyperbolic football myself, I personally find the string method smoother, more visually appealing, and more intuitive. However, the football allows some mathematically precise follow-up activities that the string method does not. For instance, you can use the football to show that triangles in this plane do not have angles adding to 180 degrees. As such, it may be highly appropriate as a follow-up activity for older students.

https://www.youtube.com/watch?v=8YikT9DtrLQ

This is a very brief video of a paraboloid someone made with Lego pieces. The attraction of this particular example is that the structure is animated, so it gives a unique perspective on the relationships between parts of the plane.

https://www.ted.com/talks/margaret_wertheim_crochets_the_coral_reef

This is a TED talk involving the intersection of fiber art, hyperbolic forms, and mathematics. It is useful in an artistic sense in that it shows another way of producing the same shape. It is also useful mathematically in that she does a conceptually complex proof in a very simple physical way using her crocheted form.